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On Pseudo-automorphisms and Fusions of an Association Scheme*

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A pseudo-automorphism of an association scheme is an automorphism of its adjacency algebra with respect to both ordinary and Hadamard multiplications. An association scheme is said to be of G-type if it admits a group G of pseudo-automorphisms acting regularly on the set of adjacency matrices that are not the identity matrix. It is shown that an association scheme of G-type is amorphous if G is an elementary abelian 2-group.

1. INTRODUCTION

Let $\mathfrak{X} = (X, \{R_i\}_{0\le i\le d})$ be a commutative association scheme of class d and $\mathfrak{A}$ the adjacency algebra of $\mathfrak{X}$. An automorphism f of $\mathfrak{X}$ falls into three categories: (1) f permutes X and fixes each R_i; (2) f permutes X and also permutes $\{R_0, R_1, \ldots, R_d\}$; (3) f is not necessarily a permutation of X but an automorphism of $\mathfrak{A}$ with respect to both ordinary and Hadamard multiplications. An automorphism of the third kind is called a *pseudo-automorphism* of $\mathfrak{X}$. Let us denote the sets of automorphisms of the first, second and third kinds by $\operatorname{Inn}\mathfrak{X}$, $\operatorname{Aut}\mathfrak{X}$ and $\operatorname{Pseu}\mathfrak{X}$, respectively. Then $\operatorname{Inn}\mathfrak{X}$ is a normal subgroup of $\operatorname{Aut}\mathfrak{X}$ and $\operatorname{Out}\mathfrak{X} = \operatorname{Aut}\mathfrak{X}/\operatorname{Inn}\mathfrak{X}$ is a subgroup of $\operatorname{Pseu}\mathfrak{X}$.

A typical example of pseudo-automorphisms comes from Galois groups. Let K be the splitting field of $\mathfrak{X}$ and L the extension of the rationals by the Krein parameters q_{ij}^k (see [13]). Then the Galois group $\operatorname{Gal}(K/L)$ acting on the set of the primitive idempotents of $\mathfrak{X}$ entrywise can be extended to automorphisms of $\mathfrak{A}$, and hence $\operatorname{Gal}(K/L)$ is isomorphic to a subgroup of $\operatorname{Pseu}\mathfrak{X}$.

A pseudo-automorphism permutes the adjacency matrices A_i $(0\le i\le d)$ and also the primitive idempotents E_i $(0\le i\le d)$. Hence there exist faithful permutation representations σ, τ of $\operatorname{Pseu}\mathfrak{X}$ into the symmetric group on the index set $\{0, 1, \ldots, d\}$ such that

$$A_i^f = A_{i^{\sigma(f)}}, \qquad E_i^f = E_{i^{\tau(f)}} \qquad (1), (2)$$

for $f \in \operatorname{Pseu}\mathfrak{X}$.

Let $P = (p_j(i))$ be the 1st eigenmatrix of $\mathfrak{X}$. Let P_0 be the right-lower $d \times d$ submatrix of P:

$$P = \begin{pmatrix} 1 & k_1 \cdots k_d \\ 1 & \\ \vdots & P_0 \\ 1 & \end{pmatrix}.$$

We call P_0 the *principal part* of the first eigenmatrix P. For the second eigenmatrix $Q = (q_j(i))$, we define the principal part Q_0 similarly. Notice that P_0 and Q_0 are

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determined up to permutations of rows and columns. With these notations, we have the following:

PROPOSITION 1.1. *Let G be a finite group of order d, and $A: G \rightarrow GL(d, \mathbf{C})$ the left regular representation of G, i.e. $(A(a))_{xy} = \delta_{x,ay}$, where $\delta_{u,v}$ is the Kronecker delta. The following conditions are equivalent to each other*:
(i) *There exists a subgroup of* Pseu $\mathfrak{X}$ *which is isomorphic to G and acts regularly on $\{A_1, \ldots, A_d\}$.*
(ii) *There exists a subgroup of* Pseu $\mathfrak{X}$ *which is isomorphic to G and acts regularly on $\{E_1, \ldots, E_d\}$.*
(iii) *$k_1 = \cdots = k_d$, where k_i is the ith valency, i.e. $k_i = p_i(0)$, and $P_0 = \sum_{a \in G} \eta_a A(a)$, i.e. $(P_0)_{xy} = \eta_a$ with $a = xy^{-1}$ by a suitable rearrangement of rows and columns.*
(iv) *$m_1 = \cdots = m_d$, where m_i is the ith multiplicity, i.e. $m_i = q_i(0)$, and $\overline{Q_0}^{\mathrm{T}} = \sum_{a \in G} \eta_a A(a)$, i.e. $(Q_0)_{xy} = \overline{\eta_a}$ with $a = yx^{-1}$ by a suitable rearrangement of rows and columns.*

If one of the four equivalent conditions of Proposition 1.1 holds, then the finite group G is said to be a *regular* subgroup of Pseu $\mathfrak{X}$, and $\mathfrak{X}$ is said to be of *G-type*. The proof of Proposition 1.1 will be given in Section 2. An association scheme $\mathfrak{X}$ of class d with the property $m_1 = \cdots = m_d$ is called pseudocyclic [4, p. 48]. However, we do not use the terminology 'pseudocyclic' in this sense, since the property $m_1 = \cdots = m_d$ has little to do with cyclic groups. For example, if Pseu $\mathfrak{X}$ is transitive on $\{E_1, E_2, \ldots, E_d\}$, then $m_1 = \cdots = m_d$ holds regardless of the structure of Pseu $\mathfrak{X}$.

Let $\{\Lambda_j\}_{0 \leq j \leq e}$ be a partition of the index set $\{0, 1, \ldots, d\}$ with $\Lambda_0 = \{0\}$. Let $R_{\Lambda_j} = \bigcup_{i \in \Lambda_j} R_i$. If $(X, \{R_{\Lambda_j}\}_{0 \leq j \leq e})$ becomes an association scheme, then it is called a *fusion scheme* of $\mathfrak{X} = (X, \{R_i\}_{0 \leq i \leq d})$. A symmetric association scheme $\mathfrak{X} = (X, \{R_i\}_{0 \leq i \leq d})$ is said to be *amorphous* if any partition $\{\Lambda_j\}_{0 \leq j \leq e}$ with $\Lambda_0 = \{0\}$ gives rise to a fusion scheme of $\mathfrak{X}$. The following theorem is easy to prove but worthwhile to state here, since it does not seem to have been published anywhere.

THEOREM 1.2. *Let $\mathfrak{X} = (X, \{R_i\}_{0 \leq i \leq d})$ be a symmetric association scheme of class d. Suppose $m_1 = \cdots = m_d$, where m_i is the ith multiplicity. Then the following conditions are equivalent to each other*:
(i) *$\mathfrak{X}$ is amorphous*;
(ii) Pseu $\mathfrak{X}$ *is isomorphic to S_d, the symmetric group of degree d*;
(iii) *the principal part of the first eigenmatrix of $\mathfrak{X}$ is a linear combination of I and J by a suitable rearrangement of rows and columns, where I is the identity matrix of size d and J is the all one matrix of size d.*

Cyclotomic schemes [6, p. 17] are examples of an association scheme which has a regular cyclic subgroup of pseudo-automorphisms. Baumert, Mills and Ward [3] determined exactly when a cyclotomic scheme is amorphous (or 'uniform' in their terminology). Compare the condition (iii) of Theorem 1.2 and that of Theorem 1(ii) in [3] (see, e.g., [5] for the connection between the eigenmatrix and Gaussian periods).

For a subgroup H of Pseu $\mathfrak{X}$, the set of $\sigma(H)$-orbits $\{\Lambda_j\}_{0 \leq j \leq e}$ gives rise to a fusion scheme of $\mathfrak{X}$ (see Lemma 2.2), where σ is the permutation representation defined by (1). From this fact, we see that if Pseu $\mathfrak{X}$ has a regular subgroup G isomorphic to an elementary abelian 2-group, then $\mathfrak{X}$ has a lot of fusion schemes. In fact, we can show the following:

MAIN THEOREM. *If G is an elementary abelian 2-group, then any association scheme of G-type is amorphous.*

The reader is refered to [2] for notations and general theory of association schemes. The notion and the terminology (in Russian) of 'amorphous' were introduced by Gol'fand and Klin in 1982. Fusion schemes are called cellular subrings [7], subschemes [1], or merging of classes [4]. In this paper, however, we use the terminology 'fusion scheme' after [9] to avoid any confusion with the different concept 'sub-association scheme' [2].

2. Proof of Proposition 1.1 and Theorem 1.2

Pseudo-automorphisms are characterized as follows:

Lemma 2.1. *Let G be a finite group and σ, τ faithful permutation representations of G into the symmetric group on $\{0, 1, \ldots, d\}$. Let G act on $\{A_0, A_1, \ldots, A_d\}$ and on $\{E_0, E_1, \ldots, E_d\}$ by $A_i^a = A_{i^{\sigma(a)}}$ and $E_i^a = A_{i^{\tau(a)}}$ $(a \in G)$. Then the action of G can be extended to pseudo-automorphisms of $\mathfrak{X}$ iff*

$$p_{j^{\sigma(a)}}(i^{\tau(a)}) = p_j(i) \tag{3}$$

for all $i, j \in \{0, 1, \ldots, d\}$, and $a \in G$.

Proof. Suppose a can be extended to a pseudo-automorphism of $\mathfrak{X}$. Apply a to

$$A_j = \sum_{i=0}^{d} p_j(i) E_i$$

Then we obtain (3).

Suppose (3) holds. Extend the action $E_i^a = E_{i^{\tau(a)}}$ linearly to $\mathfrak{A}$. Then we have $A_j^a = \sum_{i=0}^{d} p_j(i) E_i^a = \sum_{i=0}^{d} p_{j^{\sigma(a)}}(i^{\tau(a)}) E_{i^{\tau(a)}} = A_{j^{\sigma(a)}}$. Thus σ and τ have the common linear extension to $\mathfrak{A}$. □

Proof of Proposition 1.1. (i)⇔(ii) Let $S(a)$, $T(a)$ be the permutation matrix representation of Pseu $\mathfrak{X}$ corresponding to $\sigma(a)$, $\tau(a)$ ($a \in$ Pseu $\mathfrak{X}$), where σ, τ are as in (1) and (2). Then by (3), we obtain

$$T(a) P S(a)^{-1} = P \tag{4}$$

for $a \in$ Pseu $\mathfrak{X}$. In particular,

$$\sum_{a \in G} \operatorname{tr} T(a) = \sum_{a \in G} \operatorname{tr} S(a)$$

for a subgroup G of Pseu $\mathfrak{X}$. This means that $\sigma(G)$, $\tau(G)$ have the same number of orbits on $\{0, 1, \ldots, d\}$. Since 0 is fixed by $\sigma(G)$ and $\tau(G)$, we see that (i) and (ii) are equivalent.

(i)⇔(iii) Assume that (i) holds. Then σ, τ restricted on $\{1, \ldots, d\}$ are equivalent to the right regular representation of G. Therefore by (3), (iii) holds.

Assume that (iii) holds. Then (3) holds if we take σ, τ to be the right regular representation of G. So G can be imbedded into Pseu $\mathfrak{X}$.

(iii)⇔(iv) Assume that (iii) holds. Then $m_1 = \cdots = m_d$ since $m_i = |X| / \sum_{j=0}^{d} (1/k_j) |p_j(i)|^2$ (see [2. Ch. II, Theorem 4.1]). The second part follows from $q_j(i) = m_j \overline{p_i(j)} / k_i$ ([2, Ch. II, Theorem 3.5]). The converse is similar. □

There is a simple but useful criterion in terms of P for a given partition $\{\Lambda_j\}_{0 \le j \le e}$ with $\Lambda_0 = \{0\}$ to give rise to a fusion scheme (due to Bannai [1] and Muzichuk [15]): (i) $\Lambda_i' = \{\alpha' \mid \alpha \in \Lambda_i\}$ coincides with some Λ_j for all i, where $R_{\alpha'} = R_\alpha^{\mathrm{T}} = \{(y, x) \mid (x, y) \in$

$R_\alpha\}$, and (ii) there exists a partition $\{\Delta_i\}_{0\le i\le e}$ of $\{0, 1, \ldots, d\}$ with $\Delta_0 = \{0\}$ such that each (Δ_i, Λ_j) block of P has a constant row sum. The constant row sum turns out to be the (i, j) entry of the first eigenmatrix of the fusion scheme. An application of this criterion reveals the existence of a fusion scheme by merging the orbits of a group of pseudo-automorphisms:

LEMMA 2.2. *Let H be a subgroup* Pseu $\mathfrak{X}$ *and $\{\Lambda_j\}_{0\le j\le e}$ be the orbits of $\sigma(H)$ acting on the index set $\{0, 1, \ldots, d\}$. Then the partition $\{\Lambda_j\}_{0\le j\le e}$ gives rise to a fusion scheme of $\mathfrak{X}$.*

PROOF. Note that the permutation representation σ of Pseu $\mathfrak{X}$ commutes with the permutation $i \mapsto i'$. Hence condition (i) of the Bannai–Muzichuk criterion is satisfied. Let $\{\Delta_j\}_{0\le j\le e}$ be the orbits of $\tau(H)$ acting on the index set $\{0, 1, \ldots, d\}$. Then

$$\sum_{k\in\Lambda_i} p_k(l) = \sum_{k\in\Lambda_i} p_{k^{\sigma(h)}}(l^{\tau(h)}) = \sum_{k\in\Lambda_i} p_k(l^{\tau(h)})$$

for $h \in H$. Therefore condition (ii) of the Bannai–Muzichuk criterion is also satisfied. □

For amorphous association schemes, a theorem of A. V. Ivanov [10] is fundamental. Let $\mathfrak{X} = (X, \{R_i\}_{0\le i\le d})$ be an amorphous association scheme. Then the graph $\Gamma_i = (X, R_i)$ is strongly regular for each i $(i \ne 0)$. A. V. Ivanov's theorem claims that, if $d \ge 3$, either Γ_i is of Latin square type for all i $(i \ne 0)$, or Γ_i is of negative Latin square type for all i $(i \ne 0)$; namely, $|X| = n^2$ (a square), and either (i) there exist integers g_i $(1 \le i \le d)$ such that

$$k_i = g_i(n-1), \qquad \lambda_i = (g_i - 1)(g_i - 2) + n - 2, \qquad \mu_i = g_i(g_i - 1),$$

or (ii) there exist integers g_i $(1 \le i \le d)$ such that

$$k_i = g_i(n+1), \qquad \lambda_i = (g_i + 1)(g_i + 2) - n - 2, \qquad \mu_i = g_i(g_i + 1),$$

where k_i, λ_i, μ_i are the usual parameters of the strongly regular graph Γ_i (see [12]).

PROOF OF THEOREM 1.2. (i)$\Rightarrow$(iii) Suppose (i) holds. The case $d = 2$ is well-known (see, e.g., [12]). Suppose $d \ge 3$. Then by A. V. Ivanov's Theorem, each $\Gamma_i = (X, R_i)$ is either of Latin square type or of negative Latin square type. We may assume Γ_i is of Latin square type, since the same proof is valid for the negative Latin square type by changing g, n to $-g$, $-n$.

By [2, Ch. II, Theorem 4.3] and A. V. Ivanov's theorem, we see that each strongly regular graph (X, R_i) $(1 \le i \le d)$ has parameters $k = g(n-1)$, $\lambda = (g-1)(g-2) + n - 2$, $\mu = g(g-1)$ with $gd = n + 1$. Such a strongly regular graph has eigenvalues k, $n - g$, $-g$ with multiplicities 1, k, $n^2 - k - 1$, respectively. So the entries of the principal part P_0 consist of $n - g$, $-g$. The orthogonality relations [2, Ch. II, Theorem 3.5] imply that the row sum of P_0 is -1. Thus each row has $n - g$ exactly once, i.e. (iii) holds.

(iii)$\Rightarrow$(ii) Suppose (iii) holds. Then (3) holds for all $a \in S_d$ with σ, τ being the natural permutation representation. By Lemma 2.1, any element of S_d can be extended to a pseudo-automorphism.

(ii)$\Rightarrow$(i) Suppose (ii) holds. Then any partition $\{\Lambda_j\}_{0\le j\le e}$ of $\{0, 1, \ldots, d\}$ with $\Lambda_0 = \{0\}$ coincides with the $\sigma(H)$-orbits for some subgroup H of Pseu $\mathfrak{X}$. Since the $\sigma(H)$-orbits give rise to a fusion scheme by Lemma 2.2, $\mathfrak{X}$ is amorphous. □

3. Some General Results About Association Schemes of G-type

When an association scheme $\mathfrak{X}$ is of G-type, we always assume that the rows and the columns of the first eigenmatrix are arranged in such a way that Proposition 1.1(iii) holds.

Proposition 3.1. *Let $(X, \{R_i\}_{0\leq i\leq d})$ be an association scheme of G-type and H a subgroup of G. Then*:
(i) *the left coset decomposition by H gives rise to a fusion scheme of $(X, \{R_i\}_{0\leq i\leq d})$;*
(ii) *if H is normal, then the fusion scheme obtained in* (i) *is of G/H-type;*
(iii) *if a partition $\{\Lambda_j\}_{1\leq j\leq e}$ of G gives rise to a fusion scheme, then so does $\{\Lambda_j g\}_{1\leq j\leq e}$ for any $g\in G$.*

Proof. (i) Immediate from Lemma 2.2.
(ii) Suppose H is a normal subgroup of G. Then the (xH, yH) entry of the first eigenmatrix of the fusion scheme is $\sum_{a\in xy^{-1}H}\eta_a$, which is the (xH, yH) entry of the matrix

$$\sum_{bH\in G/H}\Big(\sum_{a\in bH}\eta_a\Big)\tilde{A}(bH),$$

where $\tilde{A}$ is the left regular representation of G/H. So the fusion scheme is of G/H-type.
(iii) There exists a partition $\{\Delta_i\}_{1\leq i\leq e}$ such that each (Δ_i, Λ_j) block of P has a constant row sum. Then $\{\Lambda_j g\}_{1\leq j\leq e}$ and $\{\Delta_i g\}_{1\leq i\leq e}$ are partitions of G and each $(\Delta_i g, \Lambda_j g)$ block of P has a constant row sum. □

Now suppose that G is abelian and let $(X, \{R_i\}_{0\leq i\leq d})$ be an association scheme of G-type. Let $\hat{G}$ be the character group of G. Let T be a square matrix of size $|G|$, the (a, χ) entry of which is $\overline{(1/\sqrt{|G|})\chi(a)}$ for $a\in G$, $\chi\in\hat{G}$. Then by the orthogonality relation of characters, T is a unitary matrix and diagonalizes the left regular representation A of G:

$$T^{-1}A(a)T = \operatorname{diag}(\chi(a))_{\chi\in\hat{G}}.$$

By $P_0=\sum_{a\in G}\eta_a A(a)$, we have

$$T^{-1}P_0T = \operatorname{diag}\Big(\sum_{a\in G}\eta_a\chi(a)\Big)_{\chi\in\hat{G}}. \tag{5}$$

Since the principal part of the second eigenmatrix Q is $\overline{P_0}^{\mathrm{T}}$ by Proposition 1.1(iv), the orthogonality relation $PQ=nI$ is equivalent to

$$\sum_{a\in G}\eta_a=-1 \qquad\text{and}\qquad P_0\overline{P}_0^{\mathrm{T}}=nI-kJ, \tag{6), (7}$$

where $n=1+kd=|X|$. The column of T corresponding to the principal character has entries all $1/\sqrt{d}$ and so is an eigenvector of J belonging to the eigenvalue d. Other columns of T belong to the kernel of J. Therefore, applying (5) to diagonalize (7), we obtain $|\sum_{a\in G}\eta_a|^2=n-kd=1$ for the principal character 1_G and

$$\sum_{a\in G}\eta_a\chi(a)=\varepsilon(\chi)\sqrt{n}\qquad\text{for }\chi\neq 1_G, \tag{8}$$

where $\varepsilon(\chi)$ is some complex number of modulus 1.

Solving (6) and (8) for η_a by the orthogonality relation of characters, we obtain

$$\eta_a = \frac{1}{|G|}\Big(-1 + \sqrt{n} \sum_{\chi \neq 1_G} \varepsilon(\chi)\chi(a)\Big). \tag{9}$$

Identifying G with $\hat{\hat{G}}$, we may regard an element of G as a function on $\hat{G}$. Thus ε, which is a function on $\hat{G} - \{1_G\}$, can be written as

$$\varepsilon = \frac{1}{\sqrt{n}} \sum_{a \in G} \eta_a a$$

by (8). By the orthogonality relation of the characters, we have

$$1 = -\sum_{a \neq 1} a$$

as a function on $\hat{G} - \{1_G\}$ and hence ε can be written as

$$\varepsilon = \frac{1}{\sqrt{n}} \sum_{a \neq 1} (\eta_a - \eta_1)a. \tag{10}$$

Clearly,

$$P_0 A(y) = \sum_{x \in G} \eta_{xy^{-1}} A(x) \tag{11}$$

for any $y \in G$. Also, if we define $T(\sigma)$ by $T(\sigma)_{g,h} = \delta_{g^\sigma,h}$ for an automorphism σ of G, then we have

$$T(\sigma)P_0T(\sigma)^{-1} = \sum_{x \in G} \eta_{x^\sigma} A(x). \tag{12}$$

4. Proof of Main Theorem

First we rule out the non-symmetric case.

Proposition 4.1. *There exists no non-symmetric association scheme of G-type for an elementary abelian 2-group G with $|G| \geq 4$.*

Proof. We proceed by induction on the order of G. Suppose first that G is of order 4, i.e. $G = \{(0, 0), (1, 0), (0, 1), (1, 1)\}$, and suppose that there exists a non-symmetric association scheme $\mathfrak{X}$ of G-type. Let $(\eta_{0,0}, \eta_{1,0}, \eta_{0,1}, \eta_{1,1})$ be the first row of P_0. At least one of the entries of P_0 is imaginary. By (11), we may assume that $\eta_{0,0}$ is imaginary. By (12), we may assume that $\eta_{0,0} = \overline{\eta_{0,1}}$. The fusion scheme corresponding to $H = \{(0, 0), (1, 0)\}$ is of $\mathbf{Z}_2$-type by Proposition 1.1. This fusion scheme is non-symmetric and the principal part of its first eigenmatrix is

$$\begin{pmatrix} \eta_{0,0} + \eta_{1,0} & \eta_{0,1} + \eta_{1,1} \\ \eta_{0,1} + \eta_{1,1} & \eta_{0,0} + \eta_{1,0} \end{pmatrix}.$$

Since the entries of the first column are complex conjugate, we obtain $\eta_{1,0} = \overline{\eta_{1,1}}$.

By (10), we have $\sqrt{n}\,\varepsilon = \alpha(1, 0) + \beta(0, 1) + \gamma(1, 1)$, with $\alpha = \eta_{1,0} - \eta_{0,0}$, $\beta = \eta_{0,1} - \eta_{0,0}$, $\gamma = \eta_{1,1} - \eta_{0,0}$. Notice that $\alpha - \beta = \bar{\gamma}$. Evaluation of ε by suitable characters of G yields

$$n = |\alpha - \beta - \gamma|^2 \qquad \text{and} \qquad n = |-\alpha + \beta - \gamma|^2.$$

Therefore the real part of $(\alpha-\beta)\bar{\gamma}$ is zero and

$$n=|\alpha-\beta|^2+|\gamma|^2.$$

Since $\alpha-\beta=\bar{\gamma}$, n is even, whereas $n=1+4k$. This is a contradiction.

Next suppose $G\simeq \mathbf{Z}_2^m$, $m\geqslant 3$. To any subgroup of order 2, there corresponds a fusion scheme. We claim that at least one of these fusion schemes is non-symmetric. Suppose otherwise. Then $\eta_a+\eta_b$ is real for any distinct $a, b\in G$. Then $\eta_a=\{(\eta_a+\eta_b)-(\eta_b+\eta_c)+(\eta_c+\eta_a)\}/2$ is real for any $a\in G$, which is a contradiction. Therefore there exists a non-symmetric fusion scheme of $\mathbf{Z}_2^{m-1}$-type, which is impossible by the induction hypothesis. □

By Proposition 3.1, it follows that for all $a\in G$, η_a is a real number. In what follows, we let G be an elementary abelian 2-group with $|G|\geqslant 4$. The values of characters of $\hat{G}$ are ± 1. Moreover, since the left-hand side of (8) is real, it follows that $\varepsilon(\chi)=\pm 1$ for $\chi\neq 1_G$.

By (9), we have

$$\sqrt{n}\,(\eta_a-\eta_b)=\frac{n}{|G|}\sum_{\chi\neq 1_G}\varepsilon(\chi)(\chi(a)-\chi(b))$$

for any $a, b\in G$. The left-hand side is an algebraic integer, and the right-hand side is a rational number. Therefore both are rational integers. Since $n\equiv 1 \pmod{|G|}$, $|G|$ divides $\sum_{\chi\neq 1_G}\varepsilon(\chi)(\chi(a)-\chi(b))$. Since

$$\left|\sum_{\chi\neq 1_G}\varepsilon(\chi)\chi(a)\right|\leqslant |G|-1,$$

we have

$$\sum_{\chi\neq 1_G}\varepsilon(\chi)(\chi(a)-\chi(b))\in\{0, |G|, -|G|\}.$$

If we define $\varepsilon(1_G)=1$ and set $b=1$, we have

$$(\varepsilon, a)_{\hat{G}}\in\{(\varepsilon, 1)_{\hat{G}}, (\varepsilon, 1)_{\hat{G}}+1, (\varepsilon, 1)_{\hat{G}}-1\}$$

for any $a\in G$, where $(\varepsilon, a)_{\hat{G}}=(1/|G|)\sum_{\chi\in\hat{G}}\varepsilon(\chi)\overline{\chi(a)}$.

We want to show that either ε or $-\varepsilon$ coincides with an irreducible character of $\hat{G}$ on $\hat{G}-\{1_G\}$. Our original proof was somewhat tedious. Instead, we use the following lemma due to Hiroshi Suzuki.

LEMMA (Suzuki). *Let G be a finite abelian group and let ε be a complex-valued function on G. Suppose that:*
(i) $|\varepsilon(a)|=1$ *for any non-identity element a of G;*
(ii) *there exists some $\alpha\in\mathbf{C}$ such that $(\varepsilon, \chi)_G\in\{\alpha, \alpha+1, \alpha-1\}$ for any irreducible character χ of G.*
Then $\varepsilon=\psi$ on $G-\{1_G\}$ or $\varepsilon=-\psi$ on $G-\{1_G\}$ for some irreducible character ψ of G.

PROOF. Let ρ be the character of the right regular representation of G, i.e.

$$\rho(a)=\begin{cases}|G| & \text{if } a=1,\\ 0 & \text{otherwise.}\end{cases}$$

Putting $\varepsilon^* = \varepsilon - \alpha\rho$, we have $(\varepsilon^*, \chi)_G \in \{0, \pm 1\}$, and $\varepsilon^* = \varepsilon$ on $G - \{1\}$. Let $\Phi = \{\chi \in \hat{G}; (\varepsilon^*, \chi) = 1\}$ and $\Psi = \{\chi \in \hat{G}; (\varepsilon^*, \chi) = -1\}$. Then

$$\varepsilon^* = \sum_{\chi \in \Phi} \chi - \sum_{\chi \in \Psi} \chi,$$

and so

$$(\varepsilon^*, \varepsilon^*)_G = |\Phi| + |\Psi|.$$

On the other hand,

$$\begin{aligned}(\varepsilon^*, \varepsilon^*)_G &= \frac{1}{|G|}\left(|\varepsilon^*(1)|^2 + \sum_{a \neq 1} |\varepsilon(a)|^2\right) \\ &= \frac{1}{|G|}\{(|\Phi| - |\Psi|)^2 + |G| - 1\}.\end{aligned}$$

Setting $r = |\Phi|$, $s = |\Psi|$ and $d = |G|$ $(d \geqslant r + s)$, we obtain

$$r + s = \frac{1}{d}\{(r-s)^2 + d - 1\},$$

i.e.

$$d(r + s - 1) = (r - s + 1)(r - s - 1).$$

If $r \geqslant s$, then $s = 0$, so that $r = 1$ or $d - 1$. Similarly, if $r < s$, then $r = 0$ and $s = 1$ or $d - 1$. Therefore, $\varepsilon^* = \pm\psi$ on $G - \{1\}$ for some irreducible character ψ of G. This implies the assertion. □

Now, Suzuki's Lemma applied to the group $\hat{G}$ implies that either ε or $-\varepsilon$ coincides with an irreducible character of $\hat{G}$ on $\hat{G} - \{1_G\}$. Thus, there exists an element $a_0 \in G$ such that

$$\eta_a = \frac{1}{|G|}\left\{-1 \pm \sqrt{n} \sum_{\chi \neq 1_G} \chi(a_0 a)\right\}$$

for $a \in G$. Therefore η_a $(a \in G)$ takes only two different values, and hence the association scheme is amorphous by Theorem 1.2. This completes the proof of the Main Theorem.

5. Remark

It would be interesting to know whether an association scheme of G-type is amorphous if G is an elementary abelian p-group of composite order with $p \geqslant 3$.

In the above proof, we only used the following properties of P_0:

(i) η_a $(a \in G)$ are algebraic integers;

(ii) the orthogonality relations (6) and (8) hold for P_0.

However, for the case in which $G \simeq \mathbf{Z}_3 \times \mathbf{Z}_3$, these properties are not sufficient to show that P_0 is a linear combination of I and J. Indeed, there exists a matrix P_0 of $\mathbf{Z}_3 \times \mathbf{Z}_3$ type in the sense of Proposition 1.1 which satisfies the orthogonality relations (6) and (8) and has entries that are all integral, but is not a linear combination of I and J. However, by the integrality of intersection numbers p_{ij}^k, such P_0 that we found cannot be realized by an association scheme.

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